ON THE SOLUTION OF THE PROBLEM OF EQUILIBRIUM OF AN ELASTIC HALF-DISK

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In the paper [1] the problem of a half-disk as applied to the transverse bending of a plate with mixed boundary conditions was considered; by introducing unknown additional constraints on the plate the problem was reduced to a singular integral equation of an unusual type. Herein a different method is described for solving stress-analysis problems in a half-disk, making it possible to reduce their solution directly to a system of linear algebraic equations.

1. We will carry out the proposed method for the case of an elastic half-disk pressed in a state of plane strain (or plane stress) against an absolutely rigid shape with a rectilinear base. It is assumed that the contact of the elastic body with the rigid punch occurs along the diameter of the circle, while the external forces acting on the body (and, of course, keeping it in a state of equilibrium), are distributed around the circumference of the semicircle according to a given law.

We take the radius of the semicircle o be unity and locate the elastic body and the punch in the plane $z = x + iy$ such that the elastic medium occupies the lower half of the circle with center at the origin. We will denote the curved portion of the boundary of the body by γ_1 and the straight segment by γ_0. Furthermore, we let S^- and S^+ be the lower and upper half-disks, respectively, γ_2 the upper semicircumference, and γ the total circumference $\gamma = \gamma_1 + \gamma_2$. We will use the notation of Muskhelishvili [2] for the elastic constants and the elements of the displacement and stress fields.

Then, under the conditions that there is no slip or normal displacement on the line of contact of the bodies (*), the elastostatics problem for the half-disk reduces to the search for functions $\varphi(z)$, $\psi(z)$, holomorphic in S^-, which satisfy the boundary conditions

$$\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)} = f(t) \quad \text{on } \gamma_1 \tag{1.1}$$

$$\varkappa\varphi(t) - \overline{t\varphi'(t)} - \overline{\psi(t)} = 0 \quad \text{on } \gamma_0 \tag{1.2}$$

where $f(t)$ is a given function of the point t on γ_1. In other words, we must solve the fundamental mixed boundary value problem of elasticity

*) The proposed method is in principle also applicable to other cases, provided that the coefficient of friction maintains a constant value along the line of contact.

theory for a half-disk.

2. We determine the function $\varphi(z)$ in the upper half-disk S^+, setting

$$\varkappa\varphi(z) = z\bar{\varphi}'(z) + \bar{\psi}(z) \qquad (\text{for } z \text{ in } S^+) \qquad (\bar{F}(z) = \overline{F(\bar{z})}) \tag{2.1}$$

If z is replaced by $\bar{z}$ in (2.1) (noting that z belongs to S^-) and both sides of the equality ard converted to their complex conjugates, then the function $\psi(z)$ may be determined through the function $\varphi(z)$, which is holomorphic both in S^- and S^+, in the following manner:

$$\psi(z) = \varkappa\bar{\varphi}(z) - z\varphi'(z) \qquad \text{for } z \text{ in } S^- \tag{2.2}$$

It is easy to verify that the equality (1.2) on the boundary is the condition for analytic continuation of the function $\varphi(z)$ through the cut γ_0. Hence, on the basis of (2.2), the problem (1.1),(1.2) reduces to a search for a single function $\varphi(z)$, which is holomorphic in the unit circle, and satisfies the following condition on the lower semicircumference

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\,\overline{\varphi'(t)} = f(t) \qquad (t \text{ on } \gamma_1) \tag{2.3}$$

We extend the equality (2.3) onto the upper semicircumference by replacing t in it by $\bar{t}$ to obtain

$$\varphi(\bar{t}) + \varkappa\,\varphi(t) - (t - \bar{t})\,\bar{\varphi}'(\bar{t}) = f(t) \qquad (t \text{ on } \gamma_2) \tag{2.4}$$

The preceding equality may be rewritten in the form

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\,\overline{\varphi'(t)} = f(\bar{t}) + (\varkappa - 1)\,[\varphi(\bar{t}) - \varphi(t)] + (t - \bar{t})\,[\overline{\varphi'(t)} + \bar{\varphi}'(t)]$$

and it is combined with (2.3) into the single boundary condition

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\,\overline{\varphi'(t)} = g(t) + \Phi(t) \qquad \text{on } \gamma \tag{2.5}$$

where

$$g(t) = f(t) \quad \text{on } \gamma_1, \qquad g(t) = f(\bar{t}) \quad \text{on } \gamma_2 \tag{2.6}$$

$$\Phi(t) = 0 \quad \text{on } \gamma_1, \qquad \Phi(t) = (\varkappa - 1)\,[\varphi(\bar{t}) - \varphi(t)] + (t - \bar{t})\,[\overline{\varphi'(t} + \bar{\varphi}'(t)] \quad \text{on } \gamma_2$$

In solving the boundary-value problem (2.5), the usual power-series method may now be applied. In the unit circle we put

$$\varphi(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \varphi'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} \tag{2.7}$$

Using the corresponding Fourier series expansions, we have

$$g(t) = \sum_{k=-\infty}^{\infty} A_k t^k, \qquad \Phi(t) = \sum_{k=-\infty}^{\infty} \Lambda_k t^k \qquad (t = e^{i\vartheta}) \tag{2.8}$$

The quantities A_k are known. The Fourier coefficients Λ_k of the unknown function $\Phi(t)$, however, are expressed by Formulas

$$\Lambda_n = \frac{1}{2\pi i}\int_0^{\pi} \{(\varkappa - 1)\,[\varphi(\bar{t}) - \varphi(t)] + (t - \bar{t})\,[\overline{\varphi'(t)} + \bar{\varphi}'(t)]\}\, t^{-n-1}\,dt \tag{2.9}$$

For functions to be determined we shall assume conditions which are ordinarily specified in the formulation of similar nonregular problems. These conditions are dictated mostly by physical considerations and are sufficient for confirming the uniqueness of the solution within the scope of ordinary analysis. We shall assume that the displacement field is continuous in the closed region, and that the stress components are continuous up to the boundaries, with the exception of $z = \pm 1$, at which points we allow singularities of order less than unity (cf., for example, [3], Sections 113-115). Under these conditions we can assert that the derivatives $\varphi'(z)$ and $\varphi''(z)$

are continuous in any part of the closed region which does not contain the points $z = \pm 1$, and that near these points the following estimates are valid:

$$|\varphi'(z)| < C|z \pm 1|^{-\alpha} \qquad (0 \leqslant \alpha < 1,\ C = \text{const})$$

The quantity in the braces under the integral is formed on the basis of the expansion (2.7). We obtain (2.10)

$$(\varkappa-1)\,[\varphi(\bar t) - \varphi(t)] + (t - \bar t)\,[\overline{\varphi'(t)} + \varphi'(t)] = \sum_{k=1}^{\infty} \Omega_k\,[a;\varkappa]\,(t^k - t^{-k}) + \bar a_1 (t - t^{-1})$$

$$\Omega_k\,[a;\varkappa] = -\,[(\varkappa-1)a_k + (k+2)\,\bar a_{k+2} - k\bar a_k] \tag{2.11}$$

Making use of (2.9), we have

$$\Lambda_0 = \sum_{k=1}^{\infty} \alpha_{k0}\,\Omega_k\,[a;\varkappa] \tag{2.12}$$

$$\Lambda_n = \frac{1}{2}(\operatorname{sgn} n)\,\{\bar a_1 + \Omega_{|n|}\,[a;\varkappa]\} + \sum_{k=2}^{\infty} \alpha_{k1}\Omega_k\,[a;x] \qquad (n = \pm 1) \tag{2.13}$$

$$\Lambda_n = \frac{1}{2}(\operatorname{sgn} n)\,\Omega_{|n|}\,[a;\varkappa] + \alpha_{1n}\bar a_1 + {\sum_{k=1}^{\infty}}' \alpha_{kn}\Omega_k\,[a;\varkappa] \qquad (n = \pm 2, \pm 3, \ldots) \tag{2.14}$$

where

$$\alpha_{kn} = \frac{1}{2\pi i}\left[\frac{(-1)^{k-n} - 1}{k-n} + \frac{(-1)^{k+n} - 1}{k+n}\right] \qquad \begin{pmatrix} k = 1, 2, \ldots \\ n = 0, \pm 1, \pm 2, \ldots \end{pmatrix}$$

The comma on the summation sign indicates that the value $k = |n|$ is omitted in the summation.

We form the left side of the equality (2.5) with the help of the series (2.7), obtaining

$$\varphi(t) + \varkappa\varphi(\bar t) + (t - \bar t)\,\overline{\varphi'(t)} =$$
$$= (1 + \varkappa)\,a_0 + 2\bar a_2 + \bar a_1 t + \sum_{k=1}^{\infty} a_k t^k - \sum_{k=1}^{\infty} \Omega_k\,[a;\varkappa+1]\,t^{-k} \tag{2.15}$$

The series (2.15) and (2.8) are now introduced into the boundary conditions (2.5). Equating the coefficients of t^n $(n = 0, 1, 2, \ldots)$ we obtain in consecutive order, using (2.12) to (2.14)

$$(1 + \varkappa)\,a_0 + 2\bar a_2 = A_0 + \sum_{k=1}^{\infty} \alpha_{k0}\,\Omega_k\,[a;\varkappa]$$

$$a_1 + \frac{1}{2}\bar a_1 = A_1 + \frac{1}{2}\,\Omega_1\,[a;\varkappa] + \sum_{k=2}^{\infty} \alpha_{k1}\,\Omega_k\,[a;\varkappa] \tag{2.16}$$

$$a_n = A_n + \alpha_{1n}\bar a_1 + \frac{1}{2}\,\Omega_n\,[a;\varkappa] + {\sum_{k=1}^{\infty}}' \alpha_{kn}\Omega_n\,[a;\varkappa] \qquad (n \geqslant 2)$$

The Fourier coefficients of the function $g(t)$ are connected by equalities $A_n = A_{-n}$ $(n = 1, 2, \ldots)$, as is clear from the definition (2.6) of $g(t)$.

For this reason equating coefficients of negative powers of t in (2.5) does not yield new equations (different from (2.16)).

The infinite system of linear equations (2.16) is therefore a complete system of equations for determining the unknown coefficients of the expansion (2.7). After the solution a_k $(k = 0, 1, \ldots)$ of this system is found, the function $\varphi(z)$ will give the solution of the problem, provided the corresponding series are uniformly convergent.

Having found $\varphi(z)$ in the circle, we obtain the function $\psi(z)$ from (2.2), after which all of the stresses and displacements may be found in the usual manner. In particular, for the stress Y_y defined by the Kolosov-Muskhelishvili formula

$$Y_y = \operatorname{Re}\{\varphi'(z) + \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)}\}$$

we have from (2.2)

$$Y_y = \operatorname{Re}\{\varphi'(z) + \varkappa\overline{\varphi'(z)} + (z - \bar{z})\overline{\varphi''(z)}\}$$

On the cut γ_0 this expression takes the form

$$Y_y = \operatorname{Re}\{\varphi'(z) + \varkappa\bar{\varphi}'(z)\} = \operatorname{Re}\sum_{k=1}^{\infty}(1+\varkappa)\,ka_k x^{k-1} \qquad (-1 < x < 1) \quad (2.17)$$

Introducing in (2.16) the new unknowns b_k defined by the equalities $b_0 = a_0$, $b_k = ka_k$ $(k = 1, 2, \ldots)$ and writing the system of equations in expanded form, we obtain

$$(1+\varkappa)\,b_0 = -\bar{b}_2 + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{1}{2k-1}\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + A_0$$

$$\frac{1+\varkappa}{2n-1}b_{2n-1} = \bar{b}_{2n-1} - \bar{b}_{2n+1} + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{4k}{4k^2-(2n-1)^2}\left[\frac{\varkappa-1}{2k}b_{2k} + \bar{b}_{2k+2} - \bar{b}_{2k}\right] + 2A_{2n-1}$$

$$\frac{1+\varkappa}{2n}b_{2n} = \bar{b}_{2n} - \bar{b}_{2n+2} + \frac{4}{\pi i}\frac{\bar{b}_1}{4n^2-1} + \frac{2}{\pi i}\sum_{k=1}^{\infty}\frac{4k-2}{(2k-1)^2-4n^2}\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + 2A_{2n}$$

$$(n = 1, 2, \ldots) \qquad (2.18)$$

Here and in the following, the first term $\bar{b}_1$ on the right-hand side of the second equality for $n = 1$ is to be omitted. Not dwelling on the investigation of the system of equations (2.18), we limit ourselves to one hint regarding its approximate solution. For the approximate solution we use the truncated system of linear equations

$$(1+\varkappa)\,b_0 = -\bar{b}_2 + \frac{2}{\pi i}\sum_{k=1}^{N}\frac{1}{2k-1}\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + A_0$$

$$\frac{1+\varkappa}{2n-1}b_{2n-1} = \bar{b}_{2n-1} - \bar{b}_{2n+1} + \frac{2}{\pi i}\sum_{k=1}^{N}\frac{4k}{4k^2-(2n-1)^2}\left[\frac{\varkappa-1}{2k}b_{2k} + \bar{b}_{2k+2} - \bar{b}_{2k}\right] + 2A_{2n-1}$$

$$\frac{1+\varkappa}{2n}b_{2n} = -\bar{b}_{2n+2} + \bar{b}_{2n} + \frac{4}{\pi i}\frac{\bar{b}_1}{4n^2-1} + \frac{2}{\pi i}\sum_{k=1}^{N}\frac{4k-2}{(2k-1)^2-4n^2}\times$$

$$\times\left[\frac{\varkappa-1}{2k-1}b_{2k-1} + \bar{b}_{2k+1} - \bar{b}_{2k-1}\right] + 2A_{2n} \quad (n = 1, 2, \ldots, N) \qquad (2.19)$$

Moreover, we set

$$b_{m+2} - b_m = 0 \quad \text{for } m = 2N-1, 2N, \ldots \qquad (2.20)$$

Then considering Equations (2.19) and (2.20) together, we have a system of $2N + 1$ linear equations in the first $2N + 1$ unknowns $b_0, b_1, \ldots b_{2N}$. We will take the solution of this system to be an approximate solution of the infinite system (2.18).

N o t e . The function $g(t)$ defined by the equality (2.6) will not be a regular function on the circumference of the disk even for the simplest loads. In the case of a uniform normal pressure on γ_1, for example, the derivatives of $g(t)$ will have discontinuities of the first order at the points $t = \pm 1$. Consequently, the Fourier series for the function $g(t)$ will as a rule converge slowly, hence in order to obtain a satisfactory numerical solution it would be necessary to retain a large number of equations in the truncated system (2.19).

On the other hand, in all cases of practical interest, the function $g(t)$ may be "smoothed" beforehand by considering on γ_2 not the equality (2.4), but the equality obtained from it by taking the complex conjugate. Then a smooth function $g(t)$ will be obtained, but in return the infinite system of equations obtained in place of (2.16) will have a more complicated structure. Such a modification of the boundary conditions is clearly equivalent to a certain transformation of the system (2.16). Such a transformation may sometimes prove to be feasible.

The boundary value problem (2.5) then takes the form

$$\varphi(t) + \varkappa\varphi(\bar{t}) + (t - \bar{t})\overline{\varphi'(t)} = g^*(t) + \Phi^*(t) \text{ on } \gamma \tag{2.21}$$

$$g^*(t) = f(t) \quad \text{on } \gamma_1, \qquad g^*(t) = \bar{f}(t) \quad \text{on } \gamma_2 \tag{2.22}$$

$$\Phi^*(t) = \varphi(t) - \bar{\varphi}(t) + \varkappa[\varphi(\bar{t}) - \overline{\varphi(t)}] + (t - \bar{t})[\overline{\varphi'(t)} - \varphi'(\bar{t})] \quad \text{on } \gamma_2$$
$$\Phi^*(t) = 0 \quad \text{on } \gamma_1 \tag{2.23}$$

In place of (2.9) and (2.10) we will have respectively

$$\Lambda_n^* = \frac{1}{2\pi i}\int_0^\pi \Phi^*(t)\, t^{-n-1}dt \qquad (n = 0, \pm 1, \ldots) \tag{2.24}$$

$$\Phi^*(t) = (\bar{a}_1 - a_1)\, t + \sum_{k=0}^\infty (a_k - \bar{a}_k)\, t^k - \sum_{k=0}^\infty \Omega_k^*[a; \varkappa]\, t^{-k} \tag{2.25}$$

$$\Omega_k^*[a; \varkappa] = (k+2)(a_{k+2} - \bar{a}_{k+2}) - k(a_k - \bar{a}_k) - \varkappa(a_k - \bar{a}_k) \tag{2.26}$$

We have, furthermore,

$$\frac{1}{\pi i}\int_0^\pi \Big[\sum_{k=0}^\infty \lambda_k t^k\Big] t^{-n-1}dt = \lambda_n + \frac{1}{\pi i}\sum_{k=0}^{\infty}{}' \delta_{k-n}\lambda_k \qquad (n \geqslant 0)$$

$$\frac{1}{\pi i}\int_0^\pi \Big[\sum_{k=0}^\infty \lambda_k t^{-k}\Big] t^{-n-1}dt = \lambda_{-n} - \frac{1}{\pi i}\sum_{k=0}^{\infty}{}' \delta_{k+n}\lambda_k \qquad (n \leqslant 0)$$

$$\frac{1}{\pi i}\int_0^\pi t^{-n}\, dt = \begin{cases} i/2\delta_{n-1}/\pi & (n \neq 1) \\ 1 & (n = 1) \end{cases}, \qquad \delta_\nu = \frac{(-1)^\nu - 1}{\nu}$$

On the basis of these formulas it is now possible to obtain in a similar manner as before an infinite system of linear equations for determining the unknowns a_k. In contrast to the preceding, here we need the whole system of equations obtained by equating coefficients for all the various powers of t. After elementary reductions this system may be written in the form

$$\mathrm{Re}\,\{(1+\varkappa)\, a_0 + 2a_2\} = \frac{1}{\pi}\,\mathrm{Im}\Big[\delta_1 a_1 + \sum_{k=1}^\infty \delta_k a_k + \sum_{k=1}^\infty \delta_k \Omega_k^*[a; \varkappa]\Big] + A_0^*$$

$$\mathrm{Re}\,\{2a_1\} = \frac{1}{\pi}\,\mathrm{Im}\Big[\sum_{k=1}^{\infty}{}' \delta_{k-1} a_k + \sum_{k=0}^\infty \delta_{k+1}\, \Omega_k^*[a; \varkappa]\Big] + A_1^* \tag{2.27}$$

$$\operatorname{Re}\{(\varkappa-1)a_1+3a_3\}=\frac{1}{\pi}\operatorname{Im}\left[\sum_{k=0}^{\infty}\delta_{k+1}a_k+\sum_{k=0}^{\infty}{}'\delta_{k-1}\ \Omega_k^*[a;\varkappa]\right]+A_{-1}^* \qquad \begin{matrix}(2.27)\\ \text{cont.}\end{matrix}$$

$$\operatorname{Re}\{a_n\}=\frac{1}{\pi}\operatorname{Im}\left[\delta_{n-1}a_1+\sum_{k=0}^{\infty}{}'\delta_{k-n}\ a_k+\sum_{k=0}^{\infty}\delta_{k+n}\Omega_k^*[a;\varkappa]\right]+A_n^*$$

$$\operatorname{Re}\{(n+2)a_{n+2}-(n-\varkappa)a_n\}= \qquad (n\geqslant 2)$$

$$=\frac{1}{\pi}\operatorname{Im}\left[-\delta_{n+1}a_1+\sum_{k=0}^{\infty}\delta_{k+n}\ a_k+\sum_{k=0}^{\infty}{}'\delta_{k-n}\ \Omega_k^*[a;\varkappa]\right]+A_{-n}^*$$

In these equalities A_n^* $(n=0,\pm 1,\ldots)$ denote the Fourier coefficients of the function $g^*(t)$; A_n^* are prescribed real numbers. The equalities (2.27) are an infinite system of real equations for the real and imaginary parts of a_k $(k=0,1,\ldots)$. We set $a_k=\alpha_k'+i\beta_k'$ and make the substitution

$$\alpha_0'=\alpha_0,\qquad \beta_0'=\pi\beta_0,\quad n\alpha_n'=\alpha_n,\qquad n\beta_n'=\pi\beta_n \qquad (2.28)$$

Then the system (2.27) for α_n, β_n takes the form

$$(1+\varkappa)\alpha_0+\alpha_2=-2\beta_1+\sum_{k=1}^{\infty}\frac{\delta_k}{k}\beta_k+\sum_{k=1}^{\infty}\delta_k(\beta_{k+2}-\omega_k\beta_k)+A_0^*$$

$$2\alpha_1=\sum_{k=0}^{\infty}\frac{\delta_{k-1}}{k}\ \beta_k+\sum_{k=0}^{\infty}{}'\delta_{k+1}\ (\beta_{k+2}-\omega_k\beta_k)+A_1^*\left(\omega_\nu=1+\frac{\varkappa}{\nu}\right) \qquad (2.29)$$

$$(\varkappa-1)\alpha_1+\alpha_3=\sum_{k=0}^{\infty}\frac{\delta_{k+1}}{k}\ \beta_k+\sum_{k=0}^{\infty}{}'\delta_{k-1}\ (\beta_{k+2}-\omega_k\beta_k)+A_{-1}^*$$

$$\frac{\alpha_n}{n}=\delta_{n-1}\beta_1+\sum_{k=0}^{\infty}{}'\frac{\delta_{k-n}}{k}\ \beta_k+\sum_{k=0}^{\infty}\delta_{k+n}\ (\beta_{k+2}-\omega_k\beta_k)+A_n^*$$

$$\alpha_{n+2}-\omega_{-n}\alpha_n=-\delta_{n+1}\beta_1+\sum_{k=0}^{\infty}\frac{\delta_{k+n}}{k}\beta_k+\sum_{k=0}^{\infty}{}'\delta_{k-n}(\beta_{k+2}-\omega_k\beta_k)+A_{-n}^* \qquad (n\geqslant 2)$$

Eliminating from this system α_k $(k=1,2,\ldots)$, we obtain a system for determining β_k $(k=0,1,\ldots)$, which breaks up into two independent systems in the unknowns with even and odd indices. This system takes the form

$$\omega_{01}\beta_0+\sum_{k=1}^{\infty}{}'\omega_{k1}\beta_k=B_1,\qquad \omega_{0n}\beta_0+\sum_{k=1}^{\infty}{}'\omega_{kn}\beta_k=B_n \qquad (n\geqslant 2) \qquad (2.30)$$

$$\omega_{01}=-(\varkappa^2+4\varkappa+3),\qquad \omega_{0n}=(1+\varkappa)^2\delta_n$$

$$\omega_{k1}=((-1)^{k-1}-1)\left(\frac{4k+3\varkappa+3}{k(k+3)}+\frac{(1+\varkappa)^2(k-1)+k(5-8\varkappa-5k)}{2k(k^2-1)}\right)$$

$$\omega_{kn}=((-1)^{k-n}-1)\left(\frac{(k-n)[(\varkappa+k)^2+1-kn-2k]-2\varkappa(k+n)}{k(k^2-n^2)}-\right.$$

$$\left.-\frac{(k+n)[k-n-2(1-\varkappa)]}{(k+n)^2-4}\right) \qquad (n=2,3,\ldots)$$

$$B_1={}^1/_2(\varkappa-1)A_1^*+3A_3^*-A_{-1}^*,\qquad B_n=(n+2)A_{n+2}^*-(n-\varkappa)A_n^*-A_{-n}^*$$

When β_n is determined, α_n may be found directly from (2.29) (α_0 is determined from the first equation of the system), and then the sought-for holomorphic function $\varphi(z)$ is found.

3. As a numerical example we consider the equilibrium of a half-disk pressed against a rigid punch by a uniform pressure. Then

$$f(t) = pt \qquad (t \text{ on } \gamma_1) \tag{3.1}$$

where p denotes the magnitude of the pressure. In accordance with (2.6), for $g(t)$ we have

$$g(t) = pe^{-i\vartheta} \quad (0 \leqslant \vartheta \leqslant \pi), \qquad g(t) = pe^{i\vartheta} \quad (\pi \leqslant \vartheta \leqslant 2\pi)$$

Consequently

$$\operatorname{Re}\{g(t)\} = p\cos\vartheta, \qquad \operatorname{Im}\{g(t)\} = -p\,|\sin\vartheta| \tag{3.2}$$

In the present case we have

$$A_1 = \frac{p}{2}, \qquad A_{2n-1} = 0 \quad (n \geqslant 2), \qquad A_{2n} = \frac{2}{\pi i}\frac{p}{1-4n^2} \tag{3.3}$$

The truncated system (2.19), (2.20), consisting of $4N + 2$ real equations, was numerically solved for the values $N = 3$, 4 and 5. For plane strain, it was assumed that $\varkappa = {}^5/_3$ (Poisson's ratio equal to ${}^1/_3$).

The boundary condition (2.3), which in the present case, after substitution of the corresponding series on the left-hand side, takes the form

$$(1+\varkappa)\, b_0 + \bar{b}_2 + b_1 e^{-i\vartheta} + \sum_{k=1}^{\infty}\frac{b_k}{k}e^{-ik\vartheta} + \sum_{k=1}^{\infty}\left(\frac{\varkappa}{k}b_k + \bar{b}_{k+2} - \bar{b}_k\right)e^{ik\vartheta} = pe^{-i\vartheta}$$
$$(0 \leqslant \vartheta \leqslant \pi) \tag{3.4}$$

was checked at the points $\vartheta = \vartheta_k = {}^1/_2\pi k / N \; (k = 0, 1, \ldots, N)$. The real part of this equality was satisfied by the approximate solution to a high degree of accuracy even for $N = 3$, whereas the discrepancy in its imaginary part approaches zero rather slowly, as should be expected. We cite below characteristic values of the normal stress Y_y, determined from (2.17) at particular points of the base of the punch $(-1 \leqslant x \leqslant 1)$

$$\max Y_y = Y_y|_{x=0} = 1.1591p, \qquad \min Y_y = Y_y|_{x=\pm 1} = 0.8376p \tag{3.5}$$

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